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A LINEAR ELLIPTIC SINGULAR PERTURBATION PROBLEM

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Contents

1. Introduction

In this report a singular perturbation problem for a linear partial differential equation of the elliptic type is studied. First, a short description of the problem is given.
Let

$$(1.1) \qquad \varepsilon L_2\phi + L_1\phi = h(x,y)$$

be a differential equation, where L_2 is a linear, second order differential operator and L_2 a linear, first order differential operator. Further, $h(x,y)$ is a given function and ε is a small positive parameter. We will suppose that this equation holds in a bounded strictly convex domain G and that the operator L_2 is elliptic in $\overline{G}$. Furthermore, we impose on $\phi(x,y;\varepsilon)$ the condition that along the boundary Γ of G, $\phi(x,y;\varepsilon)$ assumes prescribed values $\psi(x,y)$.
It is well-known that for this type of singular perturbation problem the asymptotic approximation contains so-called "boundary-layer" terms, which are asymptotically equivalent to zero everywhere in G except for a small neighbourhood of a part Γ_r of the boundary Γ. In this neighbourhood the derivatives of the boundary-layer terms in a direction normal to Γ_r behave as inverse powers of ε.

We know from [3], that an asymptotic approximation can be obtained by the singular perturbation method. This approximation contains terms, which are singular in both end points A and B of the part of Γ_r of the boundary. Therefore, neighbourhoods of these points are excluded in constructing the approximation.

It is the aim of this paper to construct an approximation of $\phi(x,y;\varepsilon)$, which is uniformly valid in the complete domain $\overline{G}$. The proof of the uniform validity of this approximation is given with the aid of a theorem of Eckhaus and De Jager [3]. This theorem was proved with the maximum principle for elliptic differential equations.

In order to analyse the behaviour of the function $\phi(x,y;\varepsilon)$ near A and B, a method of boundary-layer theory is applied, i.e. the coordinate stretching

technique. However, instead of stretching only the coordinate in the direction of the inner normal of Γ_r, as is usually done in boundary-layer theory, both the normal coordinate and the coordinate, which is tangent to the boundary, are stretched in A and B. It appears that two types of boundary layers near A (and B) exist, of which one seems to be unknown. We will call these boundary layers intermediate and interior boundary-layer.
The interior boundary-layer contains the point A and the corresponding expansion satisfies the prescribed boundary values. The expansion in the intermediate boundary-layer matches the asymptotic approximation of $\phi(x,y;\varepsilon)$ given in [3] (valid outside a neighbourhood of A), and the expansion in the interior boundary-layer as well.

An outline of such an approach has been given by Eckhaus [1]. We will continue from his results and give a further development of this aspect of singular perturbation theory. For a comparable study of equation (1.1) in a non-strictly convex and a non-convex domain, the reader is referred to Grasman [7] and Mauss [9], respectively.

The following publications deal with the problem in this report, Višik and Lyusternik [5], Knowless and Messick [4] and Eckhaus and De Jager [3].

In chapter 2 we study some local expansions of $\phi(x,y;\varepsilon)$ satisfying equation (1.1). Here we determine the terms of the several boundary-layer expansions. In chapter 3 we show that an asymptotic approximation of $\phi(x,y;\varepsilon)$ exists, which is uniformly valid in $\bar{G}$. Finally, in chapter 4 we consider the case that the order of tangency of the characteristics of L_1 to Γ is higher than one. It turns out that the results of chapter 2 and 3 still hold, apart from some modifications.

Summarizing the results of this study we conclude that

a. It is well-known that a boundary layer exists along a part Γ_r of the boundary. It appears, however, that near the endpoints of Γ_r still other boundary layers exist.

b. An asymptotic approximation of $\phi(x,y;\varepsilon)$, uniformly valid in $\bar{G}$, is constructed. Further, it is shown that the results of Frankena [6], who gave estimates of the accuracy of such a uniform approximation, can be improved.

2. Local asymptotic expansions

2.1. Introductory remarks

We consider the function $\phi(x,y;\varepsilon)$, satisfying a differential equation of elliptic type, i.e.

$$(2.1)\qquad L_3\phi \equiv \varepsilon L_2\phi + L_1\phi = h(x,y) \text{ in } G,$$

where L_1 and L_2 denote the differential operators

$$L_2 \equiv a(x,y)\,\frac{\partial^2}{\partial x^2} + 2b(x,y)\,\frac{\partial^2}{\partial x\partial y} + c(x,y)\,\frac{\partial^2}{\partial y^2} +$$

$$+ d(x,y)\,\frac{\partial}{\partial x} + e(x,y)\,\frac{\partial}{\partial y} + f(x,y),$$

and

$$L_1 \equiv -\frac{\partial}{\partial x} - g(x,y).$$

Let the operator L_2 be elliptic, $a(x,y) > 0$ and $g(x,y) - \varepsilon f(x,y) > 0$ in $\overline{G}$, and let the coefficients $a(x,y)$, ..., $h(x,y)$ be continuously differentiable up to order $2m + 3$ $(m=0,1,2,\ldots)$ in $\overline{G}$. Moreover, we assume that G is a bounded strictly convex domain with a smooth boundary Γ, which has the property that its parametric representation with the arc length as parameter is continuously differentiable up to order $2m + 6$. At the boundary $\phi(x,y;\varepsilon)$ satisfies the condition

$$(2.2)\qquad \phi|_\Gamma = \psi(x,y),$$

where $\psi(x,y)$ is continuously differentiable up to order $2m + 3$ for all points on Γ. Without loss of generality, we may assume that the position of the domain G with regard to the x,y-coordinate system is as follows. Let A and B be the points of the boundary Γ, where the characteristics of L_1 (the lines y = constant) are tangent to Γ. We assume that A is on the positive y-axis, see figure 1. In this chapter we deal with the case of first order tangency in both A and B.

Let the part of Γ at the left-hand side of A and B be Γ_l and the part at the right-hand side Γ_r.

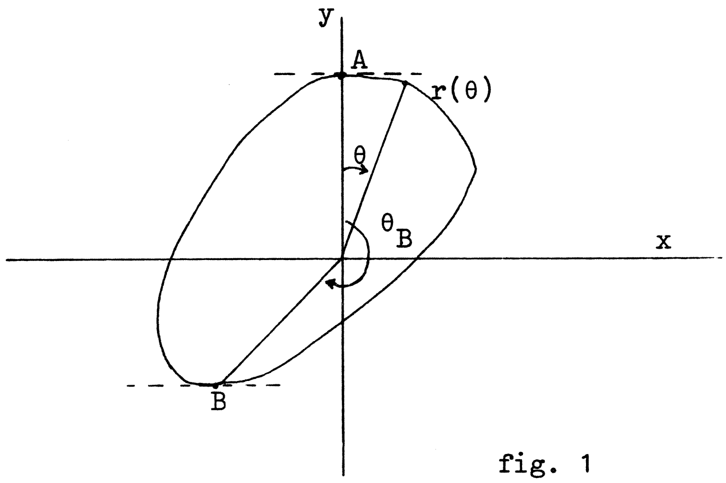

fig. 1

Problem (2.1), (2.2) belongs to the class of singular perturbation problems for $0 < \varepsilon << 1$. It is well-known that an asymptotic approximation of $\phi(x,y;\varepsilon)$ contains so called "boundary-layer" terms, which only contribute to the approximation in a small neighbourhood of Γ_r. The approximation, given in [3],does not hold in neighbourhoods of A and B. Our aim is to construct an asymptotic approximation of $\phi(x,y;\varepsilon)$, which does have the desired property.

For that purpose, we introduce the ρ,θ-coordinate system

(2.3) $\qquad x = (r(\theta)-\rho)\sin\theta, \qquad y = (r(\theta)-\rho)\cos\theta,$

where $0 \leq \theta \leq \theta_B$ and $0 \leq \rho \leq \rho_0$, see figure 1. The constant ρ_0 is chosen such that, in a sufficiently small neighbourhood, the normals at the points of Γ_r do not intersect.
Substitution of (2.3) in (2.1) yields the differential equation

(2.4) $\qquad L_\varepsilon\phi \equiv \varepsilon S_2\phi + S_1\phi = h(\rho,\theta)$

with

$$S_2 \equiv \frac{p}{(r(\theta)-\rho)^2}\frac{\partial^2}{\partial\theta^2} + \left\{\frac{2r'(\theta)p}{(r(\theta)-\rho)^2} - \frac{q}{r(\theta)-\rho}\right\}\frac{\partial^2}{\partial\theta\partial\rho} +$$

$$+ \left\{\frac{r'(\theta)^2 p}{r(\theta)-\rho)^2} - \frac{r'(\theta)q}{r(\theta)-} + t\right\}\frac{\partial^2}{\partial\rho^2} + \left\{\frac{-q}{(r(\theta)-\rho)^2} + \frac{d\cos\theta - e\sin\theta}{r(\theta)-\rho}\right\}\frac{\partial}{\partial\theta} +$$

$$+ \left\{\frac{r''(\theta)p - r'(\theta)q}{(r(\theta)-\rho)^2} - \frac{p + r(\theta)(d\cos\theta - e\sin\theta)}{r(\theta)-\rho} - (d\sin\theta + e\cos\theta)\right\}\frac{\partial}{\partial\rho} + f,$$

$$S_1 \equiv -\left[\frac{\cos\theta}{r(\theta)-\rho}\frac{\partial}{\partial\theta} + \left\{\frac{r'(\theta)\cos\theta}{r(\theta)-\rho} - \sin\theta\right\}\frac{\partial}{\partial\rho} + g\right],$$

$$p = a\cos^2\theta - 2b\sin\theta\cos\theta + c\sin^2\theta,$$

$$q = (a-c)\sin 2\theta + 2b\cos 2\theta,$$

$$t = a\sin^2\theta + 2b\sin\theta\cos\theta + c\cos^2\theta.$$

Here the coefficients a, ..., h are functions of ρ and θ.

Usually, in singular perturbation theory a coordinate stretching technique is applied by means of the local transformation $\rho = \xi\varepsilon$ in order to compute the boundary-layer terms. We introduce a more general coordinate stretching method and define the local coordinates ξ,η by

(2.5a) $\qquad \rho = \xi\varepsilon^{\nu}, \qquad \nu \geq 0,$

(2.5b) $\qquad \theta = \eta\varepsilon^{\mu}, \qquad \mu \geq 0.$

After substitution of (2.5) in (2.4) it turns out that ϕ depends on the local coordinates ξ,η and the parameter ε. This dependence is local, because ε is assumed to be small; for $\nu > 0$, $\mu = 0$ it holds near Γ_r and for $\nu > 0$, $\mu > 0$ it holds near A. In figure 2, we show some degenerations, which will appear to be important in the present problem.

In the following section we will construct four local expansions, which satisfy equation (2.4), where (2.5) is substituted with the following values of ν and μ.

a. $\nu = 0$, $\mu = 0$, the outer expansion

b. $\nu = 1$, $\mu = 0$, the boundary-layer expansion

c. $\nu = 2/3$, $\mu = 1/3$, the intermediate boundary-layer expansion

d. $\nu = 1$, $\mu = 1$, the interior boundary-layer expansion.

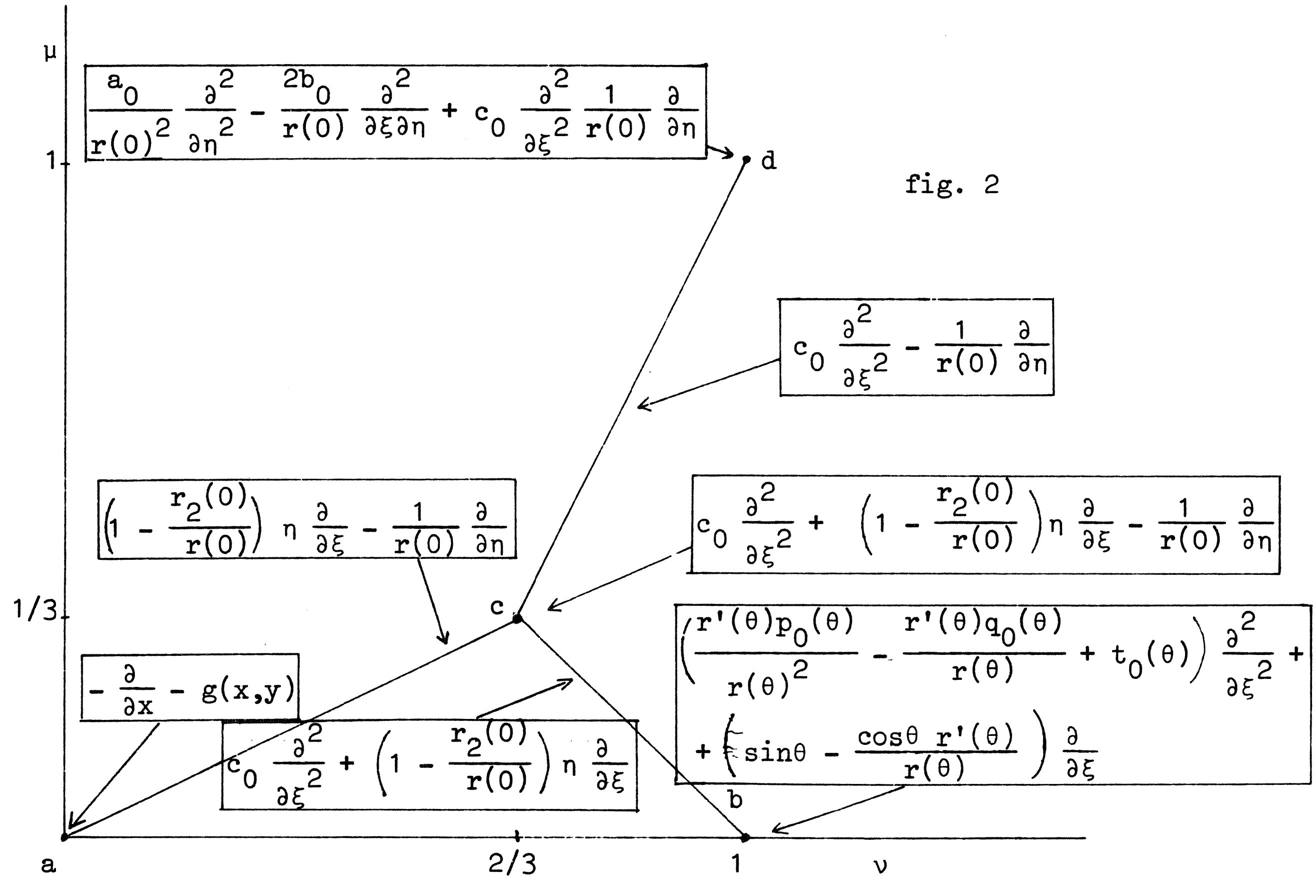

fig. 2

2.2. Asymptotic outer expansion

We assume that there exists an expansion

$$\phi_u = \sum_{n=0}^{m} U_n(x,y)\, \varepsilon^n, \tag{2.6}$$

such that the coefficients U_n satisfy the differential equations

$$L_1U_0 = h(x,y), \quad L_1U_n = L_2U_{n-1}, \quad n = 1, 2, \ldots, m.$$

Along Γ_1 the boundary conditions for $U_n(x,y)$ are

$$U_0(x,y)|_{\Gamma_1} = \psi(x,y) \quad \text{and} \quad U_n(x,y)|_{\Gamma_1} = 0, \quad n = 1, 2, 3, \ldots, m.$$

This iterative system is solved in an elementary way. It is easily verified that the functions

$$(2.7a) \qquad U_0(x,y) = \psi[k_{1y}^{-1}(y)] - \int_{k_{1x}(k_{1y}^{-1}(y))}^{x} \exp[-\int_p^x g(\bar{p},y)d\bar{p}] \;.$$

$$. \{h(p,y) + g(p,y)\, \psi[k_{1y}^{-1}(y)]\}\, dp,$$

$$(2.7b) \qquad U_n(x,y) = \int_{k_{1x}(k_{1y}^{-1}(y))}^{x} \exp[-\int_p^x g(\bar{p},y)d\bar{p}] \;.\; \{L_2U_{n-1}(p,y)\}dp,$$

$$n = 1, 2, \ldots, m,$$

satisfy both the equations and the boundary conditions. It was assumed that for $\theta_B \leq \theta \leq 2\pi$

$$x = r(\theta) \sin \theta = k_{1x}(\theta),$$

$$y = r(\theta) \cos \theta = k_{1y}(\theta),$$

and that $\quad \psi[\theta] = \psi(k_{1x}(\theta), k_{1y}(\theta)).$

2.3. Asymptotic boundary-layer expansion

Substitution of (2.5) with $\nu = 1$, $\mu = 0$ in the coefficients of equation (2.4) yields the expansions

(2.8a) $$a(\varepsilon\xi,\theta) = a_0(\theta) + \varepsilon\xi a_1(\theta) + \ldots\ldots\ldots + \varepsilon^{m+1}\ \xi^{m+1}\ a_{m+1}(\bar{\theta}),$$

. . . .,

(2.8h) $$h(\varepsilon\xi,\theta) = h_0(\theta) + \varepsilon\xi h_1(\theta) + \ldots\ldots\ldots + \varepsilon^{m+1}\ \xi^{m+1}\ a_{m+1}(\bar{\theta}),$$

(2.8i) $$p(\varepsilon\xi,\theta) = p_0(\theta) + \varepsilon\xi p_1(\theta) + \ldots\ldots\ldots + \varepsilon^{m+1}\ \xi^{m+1}\ p_{m+1}(\bar{\theta}),$$

(2.8j) $$q(\varepsilon\xi,\theta) = q_0(\theta) + \varepsilon\xi q_1(\theta) + \ldots\ldots\ldots + \varepsilon^{m+1}\ \xi^{m+1}\ q_{m+1}(\bar{\theta}),$$

(2.8k) $$t(\varepsilon\xi,\theta) = t_0(\theta) + \varepsilon\xi t_1(\theta) + \ldots\ldots\ldots + \varepsilon^{m+1}\ \xi^{m+1}\ t_{m+1}(\bar{\theta}).$$

Using these expansions we obtain for the operator L_ε

$$\varepsilon L_\varepsilon \equiv M_0 + \varepsilon M_1 + \varepsilon^2 M_2 + \ldots + \varepsilon^{m+1}\ \bar{M}_{m+1},$$

$$M_0 \equiv \left\{ \frac{r'(\theta)p_0(\theta)}{r(\theta)^2} - \frac{r'(\theta)q_0(\theta)}{r(\theta)} + t_0(\theta) \right\} \frac{\partial^2}{\partial\xi} + $$

$$+ \left\{ \sin\theta - \frac{\cos\theta\ r'(\theta)}{r(\theta)} \right\} \frac{\partial}{\partial\xi}\ .$$

We introduce the boundary-layer expansion

(2.9) $$\phi_V = \sum_{n=0}^{m+1} V_n(\xi,\theta)\ \varepsilon^n,$$

such that $V_n(\xi,\theta)$ satisfy the equations

(2.10) $$M_0V_0 = 0, \quad M_0V_n = -\sum_{i=1}^{n} M_i\ V_{n-i}, \quad n = 1, 2, \ldots, m,$$

$$M_0V_{m+1} = -\sum_{i=1}^{m} M_i\ V_{n-i} - \bar{M}_{m+1}V_0.$$

Moreover, the functions V_n satisfy at Γ_r the boundary conditions

(2.11) $$V_0(0,\theta) = \psi[\theta] - U_0(k_{3x}(\theta),k_{2y}(\theta)),\ V_n(0,\theta) = -\ U_n(k_{2x}(\theta),k_{2y}(\theta)),$$

$$n = 1, 2, \ldots, m,$$

where $k_{2x}(\theta)$ and $k_{2y}(\theta)$ denote the functions

$$x = r(\theta)\sin\theta = k_{2x}(\theta), \qquad (0 \le \theta \le \theta_B)$$
$$y = r(\theta)\cos\theta = k_{2y}(\theta),$$

and $$\psi[\theta] = \psi(k_{2x}(\theta), k_{2y}(\theta)).$$

The boundary-layer term has to vanish outside a neighbourhood of Γ_r. Therefore, we have the condition

$$\lim_{\xi\to\infty} V_n(\xi,\theta) = 0, \qquad n = 0, 1, \ldots, m.$$

We write the solutions of (2.10), which have exponential decay, as follows

(2.12) $$V_0(\xi,\theta) = A_0(\theta)\, e^{-\xi k(\theta)}, \quad V_n(\xi,\theta) = A_n(\xi,\theta)\, e^{-\xi k(\theta)},$$

$$n = 1, 2, \ldots, m+1,$$

$$k(\theta) = \left\{\sin\theta - \frac{\cos\theta\, r'(\theta)}{r(\theta)}\right\} \Big/ \left\{\frac{r'(\theta)p_0(\theta)}{r(\theta)^2} - \frac{r'(\theta)q_0(\theta)}{r(\theta)} + t_0(\theta)\right\}.$$

Using the boundary condition (2.11) we deduce that

$$A_0(\theta) = \psi[\theta] - U_0(k_{2x}(\theta), k_{2y}(\theta)).$$

For a more extensive discussion of this boundary-layer the reader is referred to [3].

2.4. Asymptotic intermediate boundary-layer expansion

In this section we consider the case that $\nu = 2/3$, $\mu = 1/3$. Again the coefficients a, ..., h are expanded in Taylor series.

(2.13a) $a(\varepsilon^{2/3}\xi,\varepsilon^{1/3}\eta)=a_0+\varepsilon^{1/3}a_{1/3}(\eta)+\varepsilon^{2/3}a_{2/3}(\xi,\eta)+\ldots+\varepsilon^{m+1}a_{(3m+3)/3}(\bar{\xi},\bar{\eta})$,

. . . . ,

(2.13h) $h(\varepsilon^{2/3}\xi,\varepsilon^{1/3}\eta)=h_0+\varepsilon^{1/3}h_{1/3}(\eta)+\varepsilon^{2/3}h_{2/3}(\xi,\eta)+\ldots+\varepsilon^{m+1}h_{(3m+1)/3}(\bar{\xi},\bar{\eta})$,

(2.13i) $p(\varepsilon^{2/3}\xi.\varepsilon^{1/3}\eta)=a_0+\varepsilon^{1/3}p_{1/3}(\eta)+\varepsilon^{2/3}p_{2/3}(\xi,\eta)+\ldots+\varepsilon^{m+1}p_{(3m+3)/3}(\bar{\xi},\bar{\eta})$,

(2.13j) $q(\varepsilon^{2/3}\xi,\varepsilon^{1/3}\eta)=2b_0+\varepsilon^{1/3}q_{1/3}(\eta)+\varepsilon^{2/3}q_{2/3}(\xi,\eta)+\ldots+\varepsilon^{m+1}q_{(3m+3)/3}(\bar{\xi},\bar{\eta})$,

(2.13k) $t(\varepsilon^{2/3}\xi,\varepsilon^{1/3}\eta)=c_0+\varepsilon^{1/3}t_{1/3}(\eta)+\varepsilon^{2/3}t_{2/3}(\xi,\eta)+\ldots+\varepsilon^{m+1}t_{(3m+3)/3}(\bar{\xi},\bar{\eta})$.

L_ε has the expansion

$$\varepsilon^{1/3}L_\varepsilon \equiv T_0 + \varepsilon^{1/3}T_1 + \varepsilon^{2/3}T_2 + \ldots + \varepsilon^{m+1}\,\bar{T}_{3m+3},$$

$$T_0 \equiv c_0\frac{\partial^2}{\partial\xi^2} + \left(1 - \frac{r_2(0)}{r(0)}\right)\eta\,\frac{\partial}{\partial\xi} - \frac{1}{r(0)}\frac{\partial}{\partial\eta}\,.$$

Assuming the existence of an intermediate boundary-layer expansion

(2.14) $$\phi_Y = \sum_{n=0}^{3m+3} Y_n(\xi,\eta)\,\varepsilon^{n/3},$$

we obtain for $Y_n(\xi,\eta)$ the equations

(2.15a) $$T_0Y_0 = 0,\quad T_0Y_n = -\sum_{i=1}^{n} T_iY_{n-i} + h_{(n-1)/3}(\xi,\eta),$$

$n = 1, 2, \ldots, 3m+2$,

(2.15b) $$T_0Y_{3m+3} = -\sum_{i=1}^{3m+2} T_iY_{3m-i+3} - \bar{T}_{3m+3}Y_0 + h_{(3m+2)/3}(\xi,\eta).$$

In addition the function $Y_n(\xi,\eta)$ has to satisfy conditions, such that it is uniquely determined. To the author's idea two different motivations will lead to solutions of the same type.

A first argument is that the intermediate boundary-layer expansion has to satisfy the following conditions. It matches the outer and the boundary-layer expansion, and has the boundary values $\psi(\varepsilon^{1/3}\eta)$ for $\xi = 0$.

A second argument differs from the first one as far as the last condition is concerned. Instead of having a prescribed value for $\xi = 0$ the intermediate boundary-layer expansion is assumed to match the interior boundary-layer expansion.

The matching principle is an important tool in singular perturbation methods. This principle is always based on a special class of singular perturbation problems. We will use the following device for the matching principle in this class of problems.

In the outer expansion (2.6) the local coordinates corresponding to $\nu = 2/3$, $\mu = 1/3$ are substituted and the expansion is reordered. This yields

$$\phi_u = U^{(0)}(\xi,\eta) + \varepsilon^{1/3}U^{(1/3)}(\xi,\eta) + \varepsilon^{2/3}U^{(2/3)}(\xi,\eta) + \ldots . \qquad (2.16)$$

The outer expansion matches the intermediate boundary-layer expansion, provided that for $\xi = C\eta^2$, $C > 0$ and $|\eta| \gg 1$,

$$Y_n(\xi,\eta) = U^{(n/3)}(\xi,\eta).$$

For this case the limit $\xi = C\eta^2$, $\eta \to \infty$ corresponds to the direction of the line, which connects a. and c., see figures 2. and 3..

The terms $U^{(n/3)}(\xi,\eta)$ consist of contributions of all terms $U_0, U_1, \ldots, U_m$ of (2.6),

$$U^{(n/3)}(\xi,\eta) = U_0^{(n/3)}(\xi,\eta) + U_1^{(n/3)}(\xi,\eta) + \ldots + U_m^{(n/3)}(\xi,\eta).$$

As a consequence of the matching principle the same expansion will hold for $Y_n(\xi,\eta)$,

$$Y_n(\xi,\eta) = Y_n^{(0)}(\xi,\eta) + Y_n^{(1)}(\xi,\eta) + Y_n^{(2)}(\xi,\eta) + \ldots + Y_n^{(m)}(\xi,\eta)$$
$$(\xi = C\eta^2 \text{ and } |\eta| \gg 1),$$

where

$$Y_n^{(k)}(\xi,\eta) \equiv U_k^{(n/3)}(\xi,\eta), \; k = 0, 1, 2, \ldots, m, \; n = 0, 1, 2, \ldots, 3m+3. \qquad (2.17)$$

It is easily deduced that $Y_n^{(k)}(\xi,\eta) = O(\eta^{n-3k})$ for $|\eta| >> 1$.

A same procedure is applied to obtain a matching condition for the boundary-layer expansion (2.9) and the intermediate boundary-layer expansion. The local coordinate corresponding to $\nu = 2/3$, $\mu = 1/3$ are substituted in expasion (2.9). Reordering of this expansion yields

$$\phi_V = \varepsilon^{1/3} \{V_0^{(1/3)}+V_1^{(1/3)}+\ldots+V_{m+1}^{(1/3)}\} + \varepsilon^{2/3} \{V_0^{(2/3)}+V_1^{(2/3)}+\ldots+V_{m+1}^{(2/3)}\}+ \ldots .$$

We observe that for $\xi = C/\eta$ $(C>0)$ and $\eta >> 1$ the function $Y_n(\xi,\nu)$ will have the following asymptotic behaviour

$$Y_n = \left\{ \bar{Y}_n^{(0)}+\hat{Y}_n^{(1)} e^{-\xi\eta \frac{r(0)-r_2(0)}{c_0 r(0)}} \right\} + \left\{ \bar{Y}_n^{(1)}+\hat{Y}_n^{(1)} e^{-\xi\eta \frac{r(0)-r_2(0)}{c_0 r(0)}} \right\} + \ldots,$$

where $\bar{Y}_n^{(k)}(\xi,\eta) \approx Y_n^{(k)}(\xi,\eta)$ and $\hat{Y}_n^{(k)}(\xi,\eta) = O(\eta^{n-3k})$. The matching condition is satisfied, if

$$\text{(2.18)} \qquad \hat{Y}_n^{(k)}(\xi,\eta)\, e^{-\xi\eta \frac{r(0)-r_2(0)}{c_0 r(0)}} \equiv V_k^{(n/3)}(\xi,\eta),\ k = 0, 1, \ldots, m+1,\quad n = 1, 2, \ldots, 3m+3.$$

According to the argument first mentioned (2.14) satisfies the condition $\phi_Y = \psi(\varepsilon^{1/3}\eta)$ at the boundary. When a Taylor expansion of this given function is made, it appears that we have the condition

$$Y_n(0,\eta) = \psi^{(n)}[0]\ \eta^n/n!$$

For $n = 0, 1$ explicite solutions of Y_n satisfying all conditions are available.

$$\text{(2.19)} \qquad Y_0(\xi,\eta) = \psi[0]$$

and

$$\text{(2.20)} \qquad Y_1(\xi,\eta) = R(\xi,\eta)\ \{\psi'[0]+r(0)\Omega\} - \eta r(0)\Omega,\ \Omega = h_0 + g_0\psi[0],$$

where $R(\xi,\eta) = -\exp(-1/2\alpha\xi\eta-1/12\beta\eta^3).\{\omega R_1+\omega^2R_2-\omega^2R_3-\omega R_4\}/\gamma.$

$$R_1(\xi,\eta) = \int_0^\infty \frac{Ai'(x)}{Ai(x)} Ai(x-m\omega\xi)\ e^{\gamma\eta x\omega^2}\ dx,$$

$$R_2(\xi,\eta) = \int_0^\infty \frac{Ai'(x)}{Ai(x)} Ai(x-m\omega^2\xi)\ e^{\gamma\eta x\omega}\ dx,$$

$$R_3(\xi,\eta) = \int_0^\infty \frac{Ai'(\ x)}{Ai(\ x)} Ai(\omega x-m\omega\xi)\ e^{\gamma\eta x}\ dx,$$

$$R_4(\xi,\eta) = \int_0^\infty \frac{Ai'(\ ^2x)}{Ai(\ ^2x)} Ai(\omega^2x-m\omega^2\xi)\ e^{\gamma\eta x}\ dx,$$

where $Ai(z)$ is the Airy function and $Ai'(z)$ its derivative,

$$\omega = \exp(2/3\pi i), \qquad \alpha = (r(0)-r_2(0))/(c_0r(0))$$

$$\omega^2 = \exp(-2/3\pi i), \qquad \beta = r(0)\alpha^2,$$

$$m^3 = \alpha/(2c_0r(0)), \qquad \gamma = c_0r(0)m^2.$$

$R(\xi,\eta)$ satisfies the homogeneous equation of (2.15) and has the boundary value $R(0,\eta) = \eta$.

We remark that the matching conditions are satisfied indeed; we verify the first two terms

$$U^{(0)}(\xi,\eta) = Y_0(\xi,\eta) = \psi[0],$$

$$U_0^{(1/3)}(\xi,\eta) = Y_1^{(0)}(\xi,\eta) = -\sqrt{\eta^2+2\xi(r(0)-r_2(0))^{-1}}\ \{\psi'[0]+r(0)\Omega\} - \eta r(0)\Omega,$$

$$V_0^{(1/3)}(\xi,\eta) = \hat{Y}_1^{(0)}(\xi,\eta)\ \exp(-\xi\eta\ \frac{r(0)-r_2(0)}{c_0r(0)}) = 2\eta\{\psi'[0]+r(0)\Omega\}\ \exp\{-\xi\eta\ \frac{r(0)-r_2(0)}{c_0r(0)}\}\ .$$

<u>Remark</u>: solution (2.20) is based on the results obtained in [2] and [8].

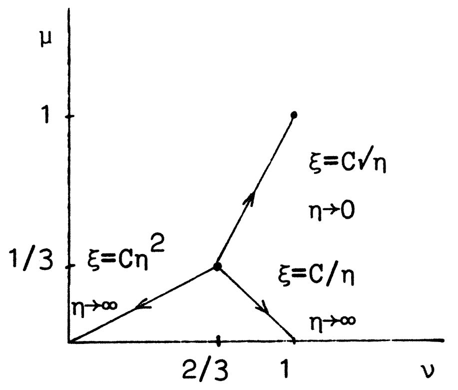

fig. 3

2.5. Asymptotic interior boundary-layer expansion

We consider the local expansion satisfying equation (2.4) in which (2.5) is substituted with $\mu = \nu = 1$. When the coefficients of this equation are expanded to ε we obtain

(2.21a) $$a(\varepsilon\xi,\varepsilon\eta) = a_0 + \varepsilon a_1(\xi,\eta) + \ldots + \varepsilon^m a_m(\bar{\xi},\bar{\eta}),$$

. . . . ,

(2.21h) $$h(\varepsilon\xi,\varepsilon\eta) = h_0 + \varepsilon h_1(\xi,\eta) + \ldots + \varepsilon^m h_m(\bar{\xi},\bar{\eta}),$$

(2.21i) $$p(\varepsilon\xi,\varepsilon\eta) = a_0 + \varepsilon p_1(\xi,\eta) + \ldots + \varepsilon^m p_m(\bar{\xi},\bar{\eta}),$$

(2.21j) $$q(\varepsilon\xi,\varepsilon\eta) = 2b_0 + \varepsilon q_1(\xi,\eta) + \ldots + \varepsilon^m q_m(\bar{\xi},\bar{\eta}),$$

(2.21k) $$t(\varepsilon\xi,\varepsilon\eta) = c_0 + \varepsilon t_1(\xi,\eta) + \ldots + \varepsilon^m t_m(\bar{\xi},\bar{\eta}).$$

The differential operator L_ε is expanded as follows

$$\varepsilon L_\varepsilon \equiv K_0 + \varepsilon K_1 + \varepsilon^2 K_2 + \ldots + \varepsilon^m \bar{K}_m,$$

$$K_0 \equiv \frac{a_0}{r(0)^2}\frac{\partial^2}{\partial\eta^2} - \frac{2b_0}{r(0)}\frac{\partial^2}{\partial\xi\partial\eta} + c_0\frac{\partial^2}{\partial\xi^2} - \frac{1}{r(0)}\frac{\partial}{\partial\eta} .$$

We assume that an expansion exists of the type

(2.22) $$\phi_W = \sum_{n=0}^{m} W_n(\xi,\eta)\, \varepsilon^n,$$

so that $W_n(\xi,\eta)$ satisfies the equations

(2.23a) $$K_0W_0 = 0, \quad K_0W_n = -\sum_{i=1}^{n} K_iW_{n-i} + h_{n-1}(\xi,\eta), \quad n = 1, 2, \ldots, m-1,$$

(2.23b) $$K_0W_m = \sum_{i=1}^{m-1} K_iW_{m-i} - \bar{K}_mW_0 + h_{m-1}(\xi,\eta),$$

and the boundary conditions

$$W_n(0,\eta) = \psi^{(n)}[0]\ \eta^n/n!$$

For n = 0, 1 we obtain the following solutions

$$W_0(\xi,\eta) = \psi[0],$$

$$W_1(\xi,\eta) = W_1^{(h)}(\xi,\eta) - \eta r(0)\Omega, \quad (\Omega = h_0+g_0\psi[0])$$

where

$$W_1^{(h)}(\xi,\eta) = \frac{ku}{\pi} \int_{-\infty}^{\infty} e^{k(v-p)}\ f_1(\frac{p\varepsilon}{r(0)})\ \frac{K_1(kr)}{r}\ dp,$$

$$u = 1/2i(\lambda_1-\lambda_2)\xi,$$

$$v = 1/2(\lambda_1+\lambda_2)\xi + r(0)\eta,$$

$$\lambda_{1,2} = (-b_0\mp i\sqrt{a_0c_0-b_0^2}\ )/C_0,$$

and $K_1(z)$ is a modified Bessel function. The function $f_1(z)$ has to be bounded and continuous, while $f_1(0) = 0$, $f_1'(0) = \{\Omega+\psi'[0]\}\ r(0)$. A function that satisfies these conditions is $f_1(z) = f_1'(0)\ ze^{-z}$.

Finally, we study the matching conditions for the interior boundary-layer expansion and the intermediate boundary-layer expansion.
In (2.22) the coordinate corresponding to $\nu = 2/3$, $\mu = 1/3$ are introduced so that

$$\phi_W = W_0^{(0)} + \varepsilon^{1/3}\{W_1^{(1/3)} + W_2^{(1/3)} + \ldots + W_m^{(1/3)}\} + \varepsilon^{2/3}\{W_1^{(2/3)} + W_2^{(2/3)} + \ldots + W_m^{(2/3)}\} + \ldots .$$

For $\eta = C\xi^2$ $(C \neq 0)$ and $0 < \xi << 1$

$$Y_n(\xi,\eta) = Y_n^{(0)}(\xi,\eta) + Y_n^{(1)}(\xi,\eta) + Y_n^{(2)}(\xi,\eta) + \ldots$$

constituted an asymptotic expansion for $Y_n(\xi,\eta)$. It appears that the matching condition is satisfied, if

$$Y_n^{(k)}(\xi,\eta) \equiv W_{(n+n_0+3k)/3}^{(n/3)} ,$$

so that $$Y_n^{(k)}(\xi,\eta) = O(\xi^{n_0+3k}),$$

$$n_0 = 1 \quad \text{for} \quad n = 2, 5, 8, \ldots$$

$$n_0 = 2 \quad \text{for} \quad n = 1, 4, 7, \ldots,$$

$$n_0 = 3 \quad \text{for} \quad n = 3, 6, 9, \ldots .$$

We remark that indeed

$$Y_1^{(0)}(\xi,\eta) = W_1^{(1/3)}(\xi,\eta) = (\eta + \frac{\xi^2}{2r(0)c_0})\ \{\psi'[0] + \Omega r(0)\} - \eta\Omega r(0).$$

3. Uniform asymptotic expansions

3.1. Introductory remarks

In this chapter we construct an expansion for the solution, which is uniformly valid in $\bar{G}$. In section 3.2 we utilize the results of the matching method to reorder the local expansions such that we obtain expansions with regular coefficients. In section 3.3 it is proved that the first three terms of a formal composite expansion approximate the solution $\phi(x,y;\varepsilon)$ with an accuracy $O(\varepsilon)$.

So far we only studied formal local expansions satisfying (2.4) in a neighbourhood of A. As we have the intention to construct a uniform approximation of the solution $\phi(x,y;\varepsilon)$, we need to investigate also local expansions satisfying (2.4) in a neighbourhood of the point B.
It can be shown by an analogous analysis that a same type of local expansions arises near B. In the sequel

$$\phi_Y = \sum_{n=0}^{3m+3} Y_n \varepsilon^{n/3}$$

represent the intermediate boundary-layer expansion near A as well as near B.

$$Y_n = Y_n^{(A)} + Y_n^{(B)},$$

where $Y_n^{(A)}(\xi,\eta)$ has the form of (2.14). $Y_n^{(B)}(\bar{\xi},\bar{\eta})$ has also such a form with the local coordinates $\bar{\xi}$, $\bar{\eta}$ given by

$$x = (r(\bar{\theta})-\bar{\rho}) \sin \bar{\theta} + x_B,$$

$$y = (r(\bar{\theta})-\bar{\rho}) \cos \bar{\theta},$$

$$\pi - \bar{\theta} = \bar{\eta}\varepsilon^{1/3}, \quad \bar{\rho} = \bar{\xi}\varepsilon^{2/3}$$

For the interior boundary-layer expansion and the matching terms a same assumption has been made.

The boundary-layer functions $V_n(\rho/\varepsilon,\theta)$ are exponentially increasing in the left part of domaim G. In order to express these boundary-layer terms as functions defined in the whole domain G we multiply them with an infinitely differentiable smoothing factor $K(\rho/\rho_0)$, which is defined in $\overline{G}$ as follows.

Outside a neighbourhood of Γ_r it equals zero, inside a neighbourhood of Γ_r we distinguish three cases for $\varepsilon^{1/3} < \theta,\ \overline{\theta} < \pi - \varepsilon^{1/3}$,

a. $0 \leq \rho \leq 1/3\ \rho_0$, where $K(\rho/\rho_0) = 1$,

b. $1/3\ \rho_0 \leq \rho \leq 2/3\ \rho_0$, where $K(\rho/\rho_0)$ is monotinic decreasing from one to zero,

c. $2/3\ \rho_0 \leq \rho \leq \rho_0$, where $K(\rho/\rho_0) = 0$.

Thus we will use in the sequel the following expression for the boundary-layer function

$$\overline{V}_n(\rho/\varepsilon,\theta) = K(\rho/\rho_0)\ V_n(\rho/\varepsilon,\theta).$$

3.2. A formal uniform asymptotic expansion

We summarize the results of the preceeding chapter as follows,

a. We have obtained an approximation of the solution, which holds outside neighbourhoods of the points A and B,

$$\phi(x,y;\varepsilon) = \sum_{n=0}^{m} \{U_n(x,y) + \overline{V}_n(\rho/\varepsilon,\theta)\}\ \varepsilon^n + Z_m(x,y;\varepsilon). \tag{3.1}$$

In [3] it is proved that $Z_m(x,y;\varepsilon) = O(\varepsilon^{m+1})$ uniformly in $\overline{G}$ apart from small neighbourhoods of A and B.

b. Near the points A and B we obtained formal local expansions, which we called intermediate and interior boundary-layer expansions.

c. With the aid of matching methods we have derived conditions for the terms of the expansions mentioned in a. and b..

In figure 4a we show the domains where a local expansion converges, the lines separating these domains are determined by the thickness (in order of magnitude of ε) of the boundary-layers, see figure 4b. The values of ν, μ of figure 4b correspond with the values of formula (2.5) (see also figure 2).

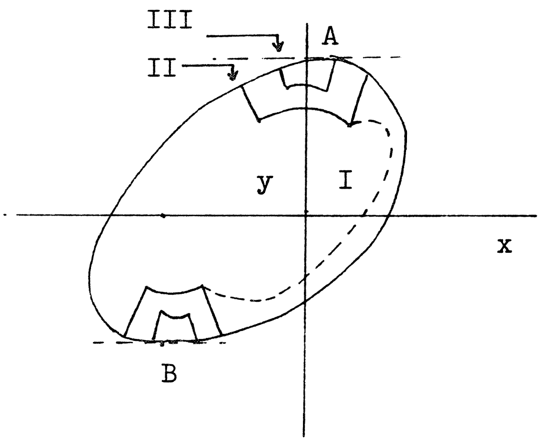

fig. 4a

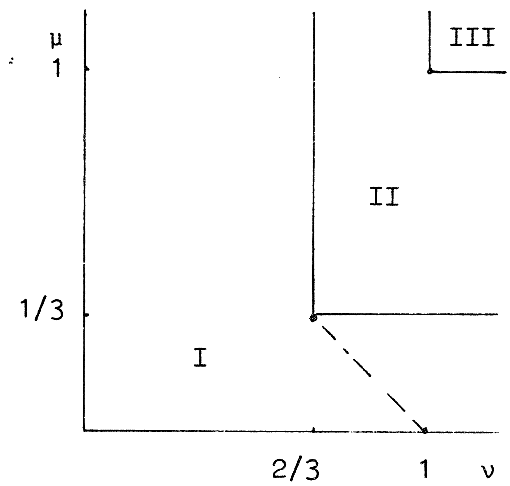

fig. 4b

Step by step a formal uniform asymptotic expansion is built up. Expansion (3.1) is reordered and the terms $U_k^{(n/3)}(x,y)$ and $\overline{V}_k^{(n/3)}(\rho/\varepsilon,\theta)$, which we obtained by the matching method, are put forwards in the series

$$\text{(3.2)} \qquad \phi = U_0 + \overline{V}_0 + \varepsilon^{1/3}(U_1^{(1/3)}+\overline{V}_1^{(1/3)}+U_2^{(1/3)}+\overline{V}_2^{(1/3)}+\ldots) +$$

$$\varepsilon^{2/3}(U_1^{(2/3)}+\overline{V}_1^{(2/3)}+U_2^{(2/3)}+\overline{V}_2^{(2/3)}+\ldots) +$$

$$\varepsilon(U_1-U_1^{(1/3)}-U_1^{(2/3)}+\overline{V}_1-\overline{V}_1^{(1/3)}-\overline{V}_1^{(2/3)}+U_2^{(3/3)}+\overline{V}_2^{(3/3)}+\ldots)+\ldots .$$

Using (2.17) and (2.18) we deduce that $Y_n(\xi,\eta)$ has the following asymptotic behaviour

$$Y_n = U_0^{(n/3)}+\overline{V}_0^{(n/3)}+U_1^{(n/3)}+\overline{V}_1^{(n/3)}+U_2^{(n/3)}+\overline{V}_2^{(n/3)}+\ldots .$$

Considering these relations we introduce an expansion, which is identical to (3.2) in domain I and to (2.14) in domain II.

$$(3.3) \quad \phi = U_0+\overline{V}_0+\varepsilon^{1/3}(Y_1-U_0^{(1/3)}-\overline{V}_0^{(1/3)})+\varepsilon^{2/3}(Y_2-U_0^{(2/3)}-\overline{V}_0^{(2/3)})+\varepsilon(U_1-U_1^{(1/3)}$$
$$- U^{(2/3)}+\overline{V}_1-\overline{V}_1^{(1/3)}-\overline{V}_1^{(2/3)}+Y_3-U_0^{(3/3)}-\overline{V}_0^{(3/3)}-U_1^{(3/3)}-\overline{V}_1^{(3/3)})+\ldots .$$

Finally we are able to construct a formal expansion, which is identical to (3.3) in I and II and to the interior boundary-layer expansion (2.22) in III.

$$(3.4) \quad \phi = U_0+\overline{V}_0+\varepsilon^{1/3}(Y_1-U_0^{(1/3)}-\overline{V}_0^{(1/3)})+\varepsilon^{2/3}(Y_2-U_0^{(2/3)}-\overline{V}_0^{(2/3)}) +$$
$$\varepsilon(U_1-U_1^{(1/3)}-U_1^{(2/3)}+\overline{V}_1-\overline{V}_1^{(1/3)}-\overline{V}_1^{(2/3)}+Y_3-U_0^{(3/3)}-\overline{V}_0^{(3/3)}-U_1^{(3/3)} -$$
$$\overline{V}_1^{(3/3)}+W_1-W_1^{(1/3)}-W_1^{(2/3)})+\ldots .$$

Expansion (3.4) has all properties desired for the function $\phi(x,y;\varepsilon)$, such as exponential decay near the boundary Γ_r and an "Airy-function" behaviour near A and B, which forms a link between the interior solution expressed in modified Bessel functions and the solution outside neighbourhoods of A and B.

However, so far we did not prove that a final number of terms of (3.4) approximates the function $\phi(x,y;\varepsilon)$ (satisfying (2.1) and (2.2)) with some accuracy in ε. This problem is investigates in section 3.3.

3.3. Application of the maximum principle

In this section we prove that

$$(3.5) \qquad = U_0+\overline{V}_0+\varepsilon^{1/3}(Y_1-U_0^{(1/3)}-\overline{V}_0^{(1/3)})+\varepsilon^{2/3}(Y_2-U_0^{(2/3)}-\overline{V}_0^{(2/3)})+$$
$$\varepsilon(\overline{V}_1-\overline{V}^{(1/3)}-\overline{V}_1^{(2/3)}+Y_3-U_0^{(3/3)}-\overline{V}_0^{(3/3)}-\overline{V}_1^{(3/3)})+Z,$$

where $Z(x,y;\varepsilon) = O(\varepsilon)$, is uniformly valid in $\overline{G}$. For that purpose a theorem of Eckhaus and De Jager [3] is reproduced, which is an application of the maximum principle. The reader will notice that in approximation (3.5) only those terms of the order $O(\varepsilon)$ of (3.4) are used, which will let cancel out the singularities in $L_\varepsilon Z$.

——— o ———

Theorem

Let $\phi(x,y;\varepsilon)$ be the solution of the boundary value problem

$$L_\varepsilon\phi = h_\varepsilon(x,y;\varepsilon)$$

valid in a bounded domain G with

$$\phi|_\Gamma = \psi_\varepsilon(x,y;\varepsilon)$$

along the boundary Γ of G. If

$$h_\varepsilon(x,y;\varepsilon) = O(\varepsilon^\alpha) \quad \text{in } \bar{G},\ \alpha \geq 0,$$

and $$\psi_\varepsilon(x,y;\varepsilon) = O(\varepsilon^\beta) \quad \text{along } \Gamma,\ \beta \geq 0,$$

then at most $$\phi\ (x,y;\varepsilon) = O(\varepsilon^{\min(\alpha,\beta)}) \text{ in } \bar{G}.$$

——— o ———

Substitution of (3.5) in (2.1) yields

$$(3.6)\quad L_\varepsilon Z = -\ [\varepsilon(L_2U_0+\bar{M}_2\bar{V}_0+\bar{M}_1\bar{V}_1+\bar{T}_3Y_1+\bar{T}_2Y_2+\bar{T}_1Y_3)+ h_0+h_1\theta+1/2h_{\theta\theta}\theta^2+h_\rho\rho-L_\varepsilon\{\varepsilon^{1/3}(U_0^{(1/3)}+\bar{V}_0^{(1/3)}+\bar{V}_1^{(1/3)})+ \varepsilon^{2/3}(U_0^{(2/3)}+V_0^{(2/3)}+V_1^{(2/3)})+\varepsilon(U_0^{(3/3)}+\bar{V}_0^{(3/3)}+\bar{V}_1^{(3/3)})\}],$$

moreover it follows from (3.5) that

$$Z|_\Gamma = O(\varepsilon).$$

The right-hand side of (3.6) contains singular terms, but it will appear that all these singularities cancel out. Our aim is to prove that the right-hadn side is of the order $O(\varepsilon)$ in $\overline{G}$. This was proved in [3] for the case, where neighbourhoods of A and B are excluded, see section 3.2. Notice that in [3] intermediate boundary-layer terms were not present in the approximation of the solution.

The following properties of the local expansion terms will be used in order to prove that for (3.6)

$$L_\varepsilon(Z) = O(\varepsilon)$$

holds uniformly in $\overline{G}$.

a. The expressions

$$K_1 = \overline{T}_3(Y_1 - U_0^{(1/3)} - \overline{V}_0^{(1/3)} - \overline{V}_1^{(1/3)}),$$
$$K_2 = \overline{T}_2(Y_2 - U_0^{(2/3)} - \overline{V}_0^{(2/3)} - \overline{V}_1^{(2/3)}),$$
$$K_3 = \overline{T}_1(Y_3 - U_0^{(3/3)} - \overline{V}_0^{(3/3)} - \overline{V}_1^{(3/3)}$$

are bounded in $\overline{G}$, so a number M independent of ε exists, such that $\max\{|K_1|,|K_2|,|K_3|\} \leq M$.

b. In the appendix we prove that

$$h_0 + h_1\theta + 1/2h_{\theta\theta}\theta^2 + h_\rho\rho +$$
$$(\varepsilon^{-1/3}T_0 + T_1 + \varepsilon^{1/3}T_2)\varepsilon^{1/3}(U_0^{(1/3)} + \overline{V}_0^{(1/3)} + \overline{V}_1^{(1/3)}) +$$
$$(\varepsilon^{-1/3}T_0 + T_1)\varepsilon^{2/3}(U_0^{(2/3)} + \overline{V}_0^{(2/3)} + \overline{V}_1^{(2/3)}) +$$
$$(\varepsilon^{-1/3}T_0)\varepsilon(U_0^{(3/3)} + \overline{V}_0^{(3/3)} + \overline{V}_1^{(3/3)}) =$$
$$\mathrm{Sing}\{L_2U_0\} + \mathrm{Sing}\{\overline{M}_2\overline{V}_0\} + \mathrm{Sing}\{\overline{M}_1\overline{V}_1\} + Z,$$

where $Z = O(\varepsilon^N)$ for $1/3\rho_0 < \rho < 2/3\rho_0$, $\varepsilon^{1/3} < \theta$, $\overline{\theta} < \pi - \varepsilon^{1/3}$ (N arbitrary large)

$Z = 0$ elsewhere in $\overline{G}$.

Sing{$S(x,y;\varepsilon)$} denotes the singular terms of a development of $S(x,y;\varepsilon)$ near A (and B). In the appendix we show that a part of the derivatives of the matching terms cancels the singular terms of the outer expansion and the boundary-layer expansion. Another part cancels the terms of the intermediate boundary-layer expansion. The contribution to the right-hand side of (3.6) comes from the regular parts of L_2U_0, $\overline{M}_2\overline{V}_0$, $\overline{M}_1\overline{V}_1$ and from K_1, K_2 and K_3. So indeed we have that $L_\varepsilon Z = O(\varepsilon)$ in $\overline{G}$.
Applying the theorem just mentioned we conclude that $Z(x,y;\varepsilon) = O(\varepsilon)$ uniformly in $\overline{G}$.

4. Higher order tangency

4.1. Introductory remarks

It appears that the order of tangency in the points A and B also determines the composition of the approximation, which is uniformly valid in $\bar{G}$. Therefore, we study this tangency in detail. In a neighbourhood of A the coordinates of the boundary $x = r(\theta)\sin\theta$, $y = r(\theta)\cos\theta$ are expanded

$$x = r(0)\theta + \left(\frac{r_2(0)}{2!} - \frac{r(0)}{3!}\right)\theta^3 + \dots,$$

$$y = r(0) + \left(\frac{r_2(0)}{2!} - \frac{r(0)}{2!}\right)\theta^2 + \frac{r_3(0)}{3!}\theta^3 + $$
$$+ \left(\frac{r_4(0)}{4!} - \frac{r_2(0)}{2!2!} + \frac{r(0)}{4!}\right)\theta^4 + \dots .$$

So far we have considered the case $r(0) > r_2(0)$, which agrees with the relation

$$x = r(0)\sqrt{\frac{2}{r(0)-r_2(0)}}\ \sqrt{r(0)-y} + O((r(0)-y))$$

between the coordinates of the boundary near A (see also Višik and Lynsternik [5]).

From now on we also consider the case, where the tangency of the characteristic of L_1 to the boundary in A is of higher order.

Let

$$y = r(0) - k_2\theta^2 - k_3\theta^3 - k_4\theta^4 - k_5\theta^5 - \dots,$$

and $k_2 = k_3 = \dots = k_{2n-1} = 0$, $k_{2n} > 0$, then the tangency is of the order $2n-1$. This leads to the following relation between the coordinates of the boundary near A

$$x = r(0)\ \sqrt[2n]{k_{2n}^{-1}}\ (r(0)-y)^{\frac{1}{2n}} .$$

An analogous argument holds for B.

4.2. Local solutions

In comparison with the local expansions given in chapter 2 we will meet an important difference with the expansions in this chapter, i.e. the intermediate boundary-layer.

First, the intermediate boundary-layer arises for other values of ν and μ, and secondly, this equation (and therefore also the local expansion) is different. Thus, the intermediate boundary-layer is determined by the order of tangency.

Let (ν_n,μ_n) be the point in the ν,μ-plane corresponding to transformation (2.5), which leads to the intermediate boundary-layer equation.

The differential operator $\varepsilon^{\mu_n}L_\varepsilon$ is expanded as follows

$$\varepsilon^{\mu_n}L_\varepsilon \equiv T_0^{(n)} + \varepsilon^{\mu_n}T_1^{(n)} + \varepsilon^{2\mu_n}T_2^{(n)} + \ldots,$$

$$T_0^{(n)} \equiv c_0\varepsilon^{2-3\nu_n+\mu_n}\frac{\partial^2}{\partial\xi^2} + \varepsilon^{\mu_n-\nu_n}X_0^{(n)} - \frac{1}{r(0)}\frac{\partial}{\partial\eta}\,.$$

We take $1-2\nu_n + \mu_n = 0$, so that the first and third term of $T_0^{(n)}$ are of the same order of magnitude in ε.

$X_0^{(n)}$ is the first term of the expanded operator

$$-\left\{\frac{r'(\eta\varepsilon^{\mu_n})\cos\eta\varepsilon^{\mu_n}}{r(\eta\varepsilon^{\mu_n}) - \xi\varepsilon^{\nu_n}} - \sin(\eta\varepsilon^{\mu_n})\right\}\frac{\partial}{\partial\xi}\,.$$

We computed for the $(2n-1)^{th}$ order of tangency the following expression for $X_0^{(n)}$

$$X_0^{(n)} \equiv k_{2n}\,\eta^{2n-1}\,\varepsilon^{(2n-1)\mu_n}\frac{\partial}{\partial\xi}\,.$$

The term $\varepsilon^{\mu_n-\nu_n}X_0^{(n)}$ is of the order $O(1)$, if $\mu_n-\nu_n + (2n-1)\mu_n = 0$. Since we also have the condition $1-2\nu_n + \mu_n = 0$, we obtain for ν_n, μ_n the values

$$\nu_n = \frac{2n}{4n-1}\,, \qquad \mu_n = \frac{1}{4n-1}\,,$$

corresponding to the $(2n-1)^{th}$ order of tangency. A diagram can be made as in figure 2. When these diagrams are brought on one figure (fig. 5), we observe that

a. $$\lim_{n\to\infty} (\nu_n,\mu_n) = (1/2,0),$$

b. in order to match the intermediate boundary-layer and the outer expansion we have to introduce the limit $\xi = C\eta^{2n}$, $|\eta| >> 1$,

c. in order to match the intermediate boundary-layer and the boudary-layer expansion we have to introduce the limit $\xi = C/\eta^{2n-1}$, $\eta >> 1$.

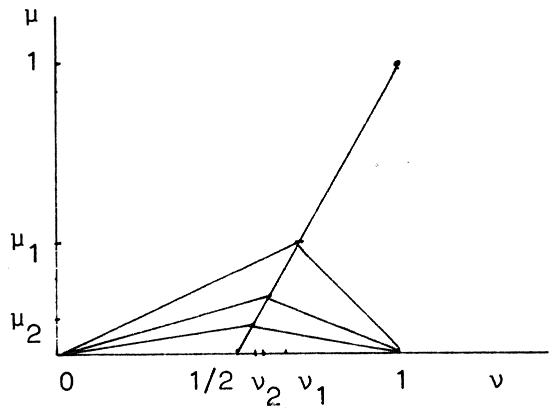

fig. 5

In the point $(\nu,\mu) = (1/2,0)$ the intermediate boundary-layer equation degenerates to a parabolic equation

$$\left(c_0 \frac{\partial^2}{\partial\xi^2} - \frac{1}{r(0)}\frac{\partial}{\partial\eta} - g_0\right) Y_1 = h_0.$$

It is easily deduced that this parabolic equation forms a local representation of a parabolic boundary-layer equation, which is obtained from (2.1) by stretching the y-coordinate, $y = r(0) - \xi_p\varepsilon^{1/2}$, and letting ε tend to zero (see figure 6).

$$\left(c(x,r(0)) \frac{\partial^2}{\partial\xi_p^2} - \frac{\partial}{\partial x} - g(x,r(0))\right) Y_p = h(x,r(0)).$$

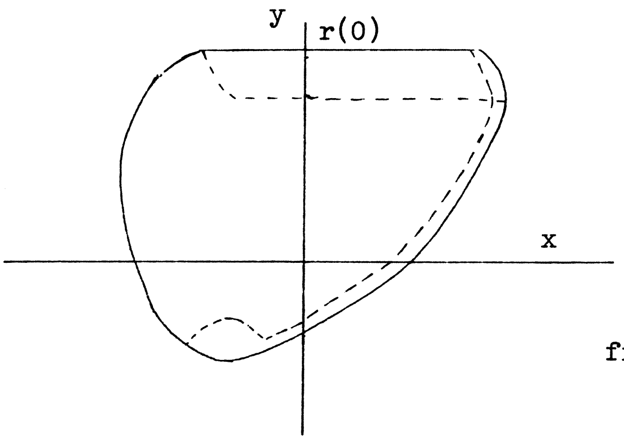

fig. 6

Returning to finite n we assume that an intermediate boundary-layer expansion exists of the type

$$\phi_Y = Y_0^{(n)} + \varepsilon^{\mu_n} Y_1^{(n)} + \varepsilon^{2\mu_n} Y_2^{(n)} + \dots .$$

Remark

Just as in the case of first order tangency a particular solution of the homogenuous intermediate boundary-layer equation is available. For example for n = 2 we have

$$\mathrm{Ai}(x-m\xi+s_2(\eta)) \; . \; \exp[-1/2\alpha\xi\eta+\gamma\eta x+t_2(\eta)],$$

$$s_2(\eta) = 1/2.k_2\eta^2 - 1/4mk_4\eta^4,$$

$$t_2(\eta) = (1/4k_2\alpha+1/12\beta)\,\eta^3 - 1/8\alpha k_4\eta^5.$$

With the aid of this particular solution we can determine $Y_k^{(n)}$, such that it satisfies all conditions.

4.3. Uniformly valid approximations

Let the order of tangency in A be $2n_A-1$ and in B $2n_B-1$. Following chapter 3 we analyse the remainder term $Z(x,y;\varepsilon)$ of

$$\phi = U_0+\bar{V}_0+\varepsilon^{\mu_A}\{Y_1^{(n_A)}-U_0^{(\mu_A)}-\bar{V}_0^{(\mu_A)}\}+\varepsilon^{\mu_B}\{Y_1^{(n_B)}-U_0^{(\mu_B)}-\bar{V}_0^{(\mu_B)}\} +$$

$$\varepsilon^{2\mu_A}\{Y_2^{(n_A)}-U_0^{(2\mu_A)}-\bar{V}_0^{(2\mu_A)}\}+\varepsilon^{2\mu_B}+\ldots+\varepsilon^{1-\mu_A}\{Y_{-1+1/\mu_A}^{(n_A)}-U_0^{(1-\mu_A)}-\bar{V}_0^{(1-\mu_A)}\} +$$

$$\varepsilon^{1-\mu_B}\{Y_{-1+1/\mu_B}^{(n_B)}-U_0^{(1-\mu_B)}-\bar{V}_0^{(1-\mu_B)}\}+\varepsilon\{\bar{V}_1-\bar{V}_1^{(\mu_A)}\ldots-\bar{V}_1^{(1-\mu_A)}-\bar{V}_1^{(\mu_B)}\ldots-\bar{V}_1^{(1-\mu_B)} +$$

$$Y_{1/\mu_A}^{(n_A)}-U_0^{(\mu_A/\mu_A)}-\bar{V}_0^{(\mu_A/\mu_A)}-\bar{V}_1^{(\mu_A/\mu_A)}+Y_{1/\mu_B}^{(n_B)}-U_0^{(\mu_B/\mu_B)}-\bar{V}_0^{(\mu_B/\mu_B)}-\bar{V}_1^{(\mu_B/\mu_B)}\} + Z.$$

The proof that $Z = 0(\varepsilon)$ uniformly in $\bar{G}$ is analogous to the proof in the case of the first order of tangency. We emphasize that an estimate of the remainder term of a truncated uniformly valid expansion of the solution can only be obtained by application of the maximum principle, if the last term of the truncated expansion contains an entire power of ε. In this last term the contribution from the interior boundary-layer expansion can be omitted.

The following statement is a consequence of the foregoing.
If the order of tangency in A is $(2n_A-1)$ and in B $(2n_B-1)$, then

$$\phi = U_0 + \bar{V}_0 + 0\left(\varepsilon^{\min(\frac{1}{4n_A-1},\frac{1}{4n_B-1})}\right),$$

uniformly in $\bar{G}$. In [6] the remainder term is estimated $0\left(\varepsilon^{\min(\frac{1}{4n_A},\frac{1}{4n_B})}\right)$.

5. Appendix

In this appendix we prove that

$$h_0 + h_1\theta + 1/2h_{\theta\theta}\theta^2 + h_\rho\rho +$$

$$(\varepsilon^{-1/3}T_0+T_1+\varepsilon^{1/3}T_2)\ \varepsilon^{1/3}(U_0^{(1/3)}+\overline{V}_0^{(1/3)}+\overline{V}_1^{(1/3)}) +$$

$$(\varepsilon^{-1/3}T_0+T_1)\ \varepsilon^{2/3}(U_0^{(2/3)}+\overline{V}_0^{(2/3)}+\overline{V}_1^{(2/3)}) +$$

$$(\varepsilon^{-1/3}T_0)\ \varepsilon(U_0^{(3/3)}+\overline{V}_0^{(3/3)}+\overline{V}_1^{(3/3)}) =$$

$$\mathrm{Sing}\{L_2U_0\} + \mathrm{Sing}\{M_2\overline{V}_0\} + \mathrm{Sing}\{\overline{M}_1\overline{V}_1\} + Z,$$

where $Z = O(\varepsilon^N)$ for $1/3\rho_0 < \rho < 2/3\rho_0$, $\varepsilon^{1/3} < \theta, \overline{\theta} < \pi - \varepsilon^{1/3}$ and N arbitrary large. Sing{S(x,y;ε)} denotes the singular terms of a development of S(x,y;ε) near A (and B), see section 3.3.

Before we prove this relation, some differential operators are introduced. Let $\rho = C\theta^2$ ($C\neq0$) and $\theta = O(\varepsilon^{1/3})$ then the operators S_1 and S_2 of (2.4) are expanded as follows

$$S_1 \equiv R_{10} + R_{11} + R_{12} + R_{13} + \dots,$$
$$S_2 \equiv R_{20} + R_{21} + R_{22} + R_{23} + \dots,$$

where

$$R_{10} \equiv -\left[\frac{1}{r(0)}\frac{\partial}{\partial\theta} + \left(\frac{r_2(0)}{r(0)} - 1\right)\theta\frac{\partial}{\partial\theta}\right],\quad R_{11} \equiv -\left[\frac{r_3(0)}{2r(0)}\theta^2\frac{\partial}{\partial\rho} + g_0\right],$$

$$R_{12} \equiv -\left[\left\{\frac{-1}{2r(0)}\left(1 + \frac{r_2(0)}{r(0)}\right)\theta^2 + \frac{\rho}{r(0)^2}\right\}\frac{\partial}{\partial\theta} + \right.$$
$$\left. + \left\{\frac{r_4(0)}{6r(0)} - \frac{r_2(0)}{2r(0)}\left(1 + \frac{r_2(0)}{r(0)} + \frac{1}{6}\right)\theta^3 + \frac{r_2(0)}{r(0)^2}\rho\theta\right\}\frac{\partial}{\partial\rho} + g_1\theta\right],$$

$$R_{20} \equiv c_0\frac{\partial^2}{\partial\rho^2},\quad R_{21} \equiv \frac{-2b_0}{r(0)}\frac{\partial^2}{\partial\rho\partial\theta} + 2b_0\left(1 - \frac{r_2(0)}{r(0)}\right)\theta\frac{\partial^2}{\partial\rho^2},$$

$$R_{22} \equiv \frac{a_0}{r(0)^2}\frac{\partial^2}{\partial\theta^2} + 2\left(\frac{a_0 r_2(0)}{r(0)^2} - \frac{a_0 - c_0}{r(0)}\right)\theta\frac{\partial^2}{\partial\rho\partial\theta} + \left\{\left(\frac{a_0 r_2(0)^2}{r(0)^2} - \frac{2r_2(0)(a_0-c_0)+r_3(0)b_0}{r(0)} + (a_0-c_0)\right)\theta^2 + c_\rho\rho\right\}\frac{\partial^2}{\partial\rho^2} +$$
$$+\left(\frac{a_0 r_2(0)}{r(0)^2} - \frac{a_0 + r(0)d_0}{r(0)} - e_0\right)\frac{\partial}{\partial\rho} .$$

Let $\xi = C/\theta$ ($C \neq 0$ and $\rho = \xi\varepsilon$) and $\theta = O(\varepsilon^{1/3})$ then the boundary-layer equation operators M_i $i = 0, 1, \ldots$ are expanded as follows

$$M_0 \equiv P_{00} + P_{01} + P_{02} + \ldots,$$
$$M_1 \equiv P_{10} + P_{11} + P_{12} + \ldots,$$
$$M_2 \equiv P_{20} + P_{21} + P_{22} + \ldots,$$
$$\ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \ldots,$$

where

$$P_{00} \equiv c_0\frac{\partial^2}{\partial\xi^2} - \left(\frac{r_2(0)}{r(0)} - 1\right)\theta\frac{\partial}{\partial\xi}, \quad P_{01} \equiv 2b_0\left(1 - \frac{r_2(0)}{r(0)}\right)\theta\frac{\partial^2}{\partial\xi^2} - \frac{r_3(0)}{2r(0)}\theta^2\frac{\partial}{\partial\xi},$$

$$P_{02} \equiv \left(\frac{a_0 r_2(0)^2}{r(0)^2} - \frac{2r_2(0)(a_0-c_0)+r_3(0)b_0}{r(0)} + (a_0-c_0)\right)\theta^2\frac{\partial^2}{\partial\xi^2} - \left\{\frac{r_4(0)}{6r(0)} + \right.$$
$$\left. - \frac{r_2(0)}{2r(0)}\left(1 + \frac{r_2(0)}{r(0)}\right) + \frac{1}{6}\right\}\theta^3\frac{\partial}{\partial\xi}, \quad P_{10} \equiv -\frac{1}{r(0)}\frac{\partial}{\partial\theta},$$

$$P_{11} \equiv -\frac{2b_0}{r(0)}\frac{\partial^2}{\partial\xi\partial\theta} - g_0, \quad P_{12} \equiv 2\left\{\frac{a_0 r_2(0)}{r(0)^2} - \frac{a_0-c_0}{r(0)}\right\}\theta\frac{\partial^2}{\partial\xi\partial\theta} + c_\xi\xi\frac{\partial^2}{\partial\xi^2} +$$
$$\left\{\frac{a_0 r_2(0)}{r(0)^2} - \frac{a_0 + r(0)d_0}{r(0)} - e_0\right\}\frac{\partial}{\partial\xi} + \frac{1}{2r(0)}\left(1 + \frac{r_2(0)}{r(0)}\right)\theta^2\frac{\partial}{\partial\theta} - \frac{r_2(0)}{r(0)^2}\xi\theta\frac{\partial}{\partial\xi} - g_1\theta,$$

$$P_{20} \equiv P_{21} \equiv 0, \quad P_{22} \equiv \frac{a_0}{r(0)^2}\frac{\partial^2}{\partial\theta^2} - \frac{\xi}{r(0)^2}\frac{\partial}{\partial\theta} .$$

It is readily established that between these operator expansions and the intermediate boundary-layer operators T_i the following relations exist:

(A1a) $$T_0 \equiv \varepsilon^{1/3}(R_{10}+\varepsilon R_{20}) \equiv \varepsilon^{-2/3}(P_{00}+\varepsilon P_{10}+\varepsilon^2 P_{20}),$$

(A1b) $$T_1 \equiv \varepsilon^{2/3}(R_{11}+\varepsilon R_{21}) \equiv \varepsilon^{-1/3}(P_{01}+\varepsilon P_{11}+\varepsilon^2 P_{21}),$$

(A1c) $$T_2 \equiv \varepsilon \quad (R_{12}+\varepsilon R_{22}) \equiv \quad (P_{02}+\varepsilon P_{12}+\varepsilon^2 P_{22}).$$

Since $S_1U_0 = h(\rho,\theta)$, we have for ρ and θ sufficiently small the relation

$$(R_{10}+R_{11}+R_{12}+\dots)\ (\psi_0+U_0^{(1/3)}+U_0^{(2/3)}+\dots) =$$
$$h_0 + h_1\theta + 1/2h_{\theta\theta}\theta^2 + h_\rho\rho + \dots .$$

Equalization of terms of a same order of magnitude yields

$$R_{10}\psi_0 = 0$$
$$R_{11}\psi_0+R_{10}\varepsilon^{1/3}U_0^{(1/3)} = h_0$$
$$R_{12}\psi_0+R_{11}\varepsilon^{1/3}U_0^{(1/3)}+R_{10}\varepsilon^{2/3}U_0^{(2/3)} = h_1\theta$$
$$R_{13}\psi_0+R_{12}\varepsilon^{1/3}\psi_0+R_{11}\varepsilon^{2/3}U_0^{(2/3)}+R_{10}\varepsilon U_0^{(3/3)} = 1/2h_{\theta\theta}\theta^2+h_\rho\rho$$

+

(A2) $$(R_{10}+R_{11}+R_{12}+R_{13})\psi_0+(R_{10}+R_{11}+R_{12})\varepsilon^{1/3}U_0^{(1/3)}+(R_{10}+R_{11})\varepsilon^{2/3}U_0^{(2/3)}+R_{10}\varepsilon U_0^{(3/3)} =$$
$$h_0 + h_1\theta + 1/2h_{\theta\theta}\theta^2 + h_\rho\rho.$$

Analogously, because $M_0V_0 = 0$ and $M_0V_1 + M_1V_1 = 0$, we have the relations

(A3a) $$(P_{00}+P_{01}+P_{02})\varepsilon^{1/3}V_0^{(1/3)}+(P_{00}+P_{01})\varepsilon^{2/3}V_0^{(2/3)}+P_{00}\varepsilon V_0^{(3/3)}= 0$$

(A3b) $$(P_{00}+P_{01}+P_{02})\varepsilon^{-2/3}V_1^{(1/3)}+(P_{00}+P_{01})\varepsilon^{-1/3}V_1^{(2/3)}+P_{00}V_1^{(3/3)} +$$
$$(P_{10}+P_{11}+P_{12})\varepsilon^{1/3}V_0^{(1/3)}+(P_{10}+P_{11})\varepsilon^{2/3}V_0^{(2/3)}+P_{10}\varepsilon V_0^{(3/3)}= 0.$$

Formulae (A1), (A2) and (A3) are utilized in order to show that

(A4) $$h_0 + h_1\theta + 1/2h_{\theta\theta}\theta^2 + h_\rho\rho +$$
$$(\varepsilon^{-1/3}T_0+T_1+\varepsilon^{1/3}T_2)\varepsilon^{1/3}(U_0^{(1/3)}+\overline{V}_0^{(1/3)}+\overline{V}_1^{(1/3)}) +$$
$$(\varepsilon^{-1/3}T_0+T_1)\varepsilon^{2/3}(U_0^{(2/3)}+\overline{V}_0^{(2/3)}+\overline{V}_1^{(2/3)}) +$$
$$(\varepsilon^{-1/3}T_0)\varepsilon(U_0^{(3/3)}+\overline{V}_0^{(3/3)}+\overline{V}_1^{(3/3)}) =$$
$$(R_{20}+R_{21}+R_{22})\varepsilon^{1/3}U_0^{(1/3)}+(R_{20}+R_{21})\varepsilon^{2/3}U_0^{(2/3)}+R_{20}\varepsilon U_0^{(3/3)} +$$
$$(P_{20}+P_{21}+P_{22})\varepsilon^{4/3}\overline{V}_0^{(1/3)}+(P_{20}+P_{21})\varepsilon^{5/3}\overline{V}_0^{(2/3)}+P_{20}\varepsilon^{2}\overline{V}_0^{(3/3)} +$$
$$(P_{10}+P_{11}+P_{12})\varepsilon^{1/3}\overline{V}_1^{(1/3)}+(P_{10}+P_{11})\varepsilon^{2/3}\overline{V}_1^{(2/3)}+P_{10}\varepsilon\overline{V}_1^{(3/3)} + Z,$$

where $Z = O(\varepsilon^N)$ for $1/3\rho_0 \leq \rho \leq 2/3\rho_0$, $\varepsilon^{1/3} < \theta$, $\overline{\theta} < \pi - \varepsilon^{1/3}$,

$Z = 0$ elsewhere in $\overline{G}$.

The right-hand side of (A4) cancels the singular terms of L_2U_0, $\overline{M}_2\overline{V}_0$ and $\overline{M}_1\overline{V}_1$.

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